

THE ZERO LOCUS OF AN ADMISSIBLE NORMAL FUNCTION

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ABSTRACT. We prove that the zero locus of an admissible normal function over an algebraic parameter space S is algebraic in the case where S is a curve.

1. INTRODUCTION

Let S be a smooth, complex projective variety. Following Morihiko Saito [8], we define an admissible normal function on S to be an admissible variation of graded-polarized mixed Hodge structure \mathcal{V} on the complement of a normal crossing divisor $D \subseteq S$ which is an extension of the trivial variation $\mathbb{Z}(0)$ by a variation of pure, polarized Hodge structure \mathcal{H} of weight $w < 0$. That is an admissible normal function is an element $\nu \in \text{Ext}_{\text{AV}}^1(\mathbb{Z}(0), \mathcal{H})$ where AV denotes the abelian category of admissible variations of mixed Hodge structure on S which are smooth on $S - D$.

Henceforth, we assume that $w = -1$. In this case, an admissible normal function corresponds to the usual notion of a horizontal normal function on S with moderate growth along D together with existence of a suitable relative weight filtration along each irreducible component of D . In this article (Theorem 4.5), we settle the following conjecture communicated to us by M. Green and P. Griffiths in the case where S is a curve.

Conjecture 1.1. *Let ν be an admissible normal function on S . Then the zero locus \mathcal{Z} of ν is an algebraic subvariety of S .*

In analogy with [1], an unconditional proof of this conjecture provides indirect evidence in support of standard conjectures on higher regulators and filtrations on Chow groups.

A rough outline of our proof is as follows: Let \mathcal{U} be an open subset of S in the analytic topology which does not intersect D . Then the zero locus of ν on \mathcal{U} is complex analytic since the restriction of ν to \mathcal{U} is a holomorphic section of associated bundle of intermediate Jacobians.

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Thus, in order to prove that the zero locus of ν is algebraic, it is sufficient to show that:

- (*) For each point $p \in D$ there exists an analytic open neighborhood $\mathcal{U}_p \subset S$ of p on which the zero locus of ν has only finitely many components.

In the case where S is a curve, we verify (*) using the orbit theorems of the second author and results of P. Deligne.

The canonical real grading $Y(s)$ (described below) of the Hodge structure \mathcal{V}_s at a point $s \in S - D$ will play a crucial role in our proof. The central idea is that ν is 0 at s if and only if $Y(s)$ is integral. It is therefore crucial to understand the asymptotics of $Y(s)$ as s tends to a point $s_0 \in D$. In Theorem 3.5, we use the SL_2 -orbit theorem of [7] to show that $Y^\dagger := \lim_{s \rightarrow s_0} Y(s)$ exists for $s_0 \in D$. Now, it is clear that ν can only vanish in a neighborhood of s_0 if Y^\dagger is non-integral. Knowing that the limit exists allows us to concentrate on the case where Y^\dagger is not integral. This case can then be handled by a rather explicit computation of the zero locus in the neighborhood of s_0 .

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2. THE ZERO LOCUS AT A SMOOTH POINT

As a preliminary step in our analysis of the zero locus of ν at infinity, we derive the local defining equations of \mathcal{Z} at an interior point of S . To this end, we begin with a review of mixed Hodge structures and their gradings.

Gradings. Let V be a finite dimensional vector space over a field k . A grading for V is a direct sum decomposition $V = \bigoplus V_k$ of V into subspace V_k indexed by integers. An n -grading of V is a direct sum decomposition $V = \bigoplus V_w$ indexed by n -tuples of integers. It is well-known (and easy to see) that gradings are in one-one correspondence with linear actions of the multiplicative group \mathbb{G}_m on V , and n -gradings are in one-one correspondence with linear actions of the n -torus \mathbb{G}_m^n on V : To an n -grading $V = \bigoplus V_w$ one associates the action where $(t_1, \dots, t_n)v = t_1^{w_1} \cdots t_n^{w_n}v$. Conversely, to an action of \mathbb{G}_m^n on V one obtains a grading by writing V as a direct sum of its isotypical subspaces.

If k is a field of characteristic 0, then gradings are in one-one correspondence with semi-simple endomorphisms Y of V with integral eigenvalues. The correspondence is the one that takes a \mathbb{G}_m action on V to its derivative at $1 \in \mathbb{G}_m$ viewed as an endomorphism of V . Conversely, it takes an endomorphism Y to the direct sum decomposition $V = \bigoplus_{k \in \mathbb{Z}} V_{Y,k}$ where $V_{Y,k} = \{v \in V \mid Yv = kv\}$. Similarly, n -gradings are in one-one correspondence with n -tuples of commuting semi-simple endomorphisms Y_1, \dots, Y_n with integral eigenvalues.

Throughout this paper, the vector spaces V which will occur will be over fields of characteristic 0. We will work with the first notion and the last notion of a grading interchangeably.

Suppose that V is equipped with a filtration

$$(2.1) \quad V = W_r V \supset W_{r-1} V \supset \cdots \supset W_{l-1} V = \{0\}$$

by subspaces W_i with $i \in \mathbb{Z}$. A *grading* of W_\bullet is then a grading $V = \bigoplus_{i \in \mathbb{Z}} V_i$ of V such that $W_k = \bigoplus_{i \leq k} V_i$.

Deligne's Grading. We now recall a fundamental result of Deligne concerning mixed Hodge structures. (See [3, Lemme 1.2.8] for the original theorem or [2, Theorem 2.13] where the result appears in the notation used below.)

A mixed Hodge structure (F, W) induces a functorial bigrading

$$(2.2) \quad V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

such that

- (1) $F^p = \bigoplus_{a \geq p} I^{a,b}$;
- (2) $W_k = \bigoplus_{a+b \leq k} I^{a,b}$;
- (3) $\bar{I}^{p,q} = I^{q,p} \bmod \bigoplus_{r < q, s < p} I^{r,s}$.

In particular, a mixed Hodge structure (F, W) induces a grading of W via the semisimple endomorphism $Y : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ which acts as multiplication by $p + q$ on $I^{p,q}$. We will call this grading *Deligne's grading* $Y_{F,W}$.

Normal functions. Returning now to the normal function setting, let S be a complex manifold of dimension n and \mathcal{V} be the variation of mixed Hodge structure associated to ν . Let $p \in \mathbb{Z}$ and (s_1, \dots, s_n) be local holomorphic coordinates on a polydisk $\Delta^n \subseteq S$ which vanish at p . Then, since Δ^n is simply connected, we can parallel translate the data of \mathcal{V} back the reference fiber $V = \mathcal{V}_p$. The Hodge filtration \mathcal{F} of \mathcal{V} then corresponds to a holomorphic, horizontal decreasing filtration

$F(s)$ of $V_{\mathbb{C}}$. The weight filtration \mathcal{W} of \mathcal{V} corresponds to a constant filtration W of $V_{\mathbb{Z}}$ with weight graded-quotients

$$\mathrm{Gr}_0^W(V_{\mathbb{Z}}) = \mathbb{Z}(0), \quad \mathrm{Gr}_{-1}^W(V_{\mathbb{Z}}) = H_{\mathbb{Z}}$$

and $\mathrm{Gr}_k^W = 0$ for $k \neq 0, -1$. Similarly, the graded-polarizations of \mathcal{W} correspond to constant polarizations of Gr^W .

On account of the short length of W , the grading

$$(2.3) \quad Y(s) = Y_{(F(s), W)}$$

defined by (2.2) can be characterized as the unique real grading of W which preserves $F(s)$. If $Y_{\mathbb{Z}}$ is any integral grading of W then the image $1 \in \mathbb{Z}(0)$ under the map

$$Y(s) - Y_{\mathbb{Z}} : \mathbb{Z}(0) \rightarrow H_{\mathbb{R}}/H_{\mathbb{Z}}$$

gives the point in the Griffiths intermediate Jacobian corresponding to the extension (2.2) via the standard isomorphism

$$H_{\mathbb{R}}/H_{\mathbb{Z}} \cong \frac{H_{\mathbb{C}}}{F^0 H_{\mathbb{C}} + H_{\mathbb{Z}}}.$$

Accordingly, p belongs to the zero locus of ν if and only if $Y(p)$ is an integral grading of W . Consequently, since $Y(s)$ is real analytic¹ in s and the set of integral gradings of W is a discrete subset of the affine space of \mathbb{R} -gradings of W , there exists a neighborhood of p in which the zero locus of ν is given by the equation

$$Y(s) = Y(p).$$

The filtration $F(s)$ takes its values in a classifying space \mathcal{M} of graded-polarized mixed Hodge structure [5]. Let $G_{\mathbb{C}}$ denote the Lie group consisting of all automorphisms of $V_{\mathbb{C}}$ which preserve W and act by complex isometries on Gr^W . Then, for each point $F \in \mathcal{M}$ there exists a neighborhood $U_{\mathbb{C}}$ of zero in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that the map

$$(2.4) \quad u \mapsto e^u.F$$

is a holomorphic submersion from $U_{\mathbb{C}}$ onto a neighborhood of F in \mathcal{M} . Accordingly, if

$$(2.5) \quad \mathfrak{g}_{\mathbb{C}} = \bigoplus_{p+q \leq 0} \mathfrak{g}^{p,q}$$

¹The grading $Y(s)$ is holomorphic when viewed as a section of the bundle of intermediate Jacobians

denotes the Deligne bigrading of the induced mixed Hodge structure $(F\mathfrak{g}_{\mathbb{C}}, W\mathfrak{g}_{\mathbb{C}})$ then map (2.3) restricts to a biholomorphism from a neighborhood of zero in subalgebra

$$q_F = \bigoplus_{p < 0, p+q \leq 0} \mathfrak{g}^{p,q}$$

onto a neighborhood of F in \mathcal{M} .

Letting $F = F(p)$, the constructions of the previous paragraph show that near p we can write

$$F(s) = e^{\Gamma(s)}.F$$

relative to a unique holomorphic function $\Gamma(s)$ with values in q_F which vanishes at p . Let $Y = Y(p)$ and

$$\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$$

denote the decomposition of $\Gamma(s)$ according to the eigenvalues of Y . By the Campbell–Baker–Hausdorff formula, there exists a universal power series $\Psi(t)$ such that

$$e^{u+v}e^{-u} = e^{\Psi(\text{ad } u)v}.$$

In particular,

$$\begin{aligned} e^{\Gamma(s)}.Y &= e^{\Gamma_0(s)+\Gamma_{-1}(s)}e^{-\Gamma_0(s)}e^{\Gamma_0(s)}.Y \\ &= e^{\Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s)}.Y = Y + \Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) \end{aligned}$$

is a holomorphic grading of the weight filtration (over \mathbb{C}) which preserves $F(s)$. Therefore, there exists a real analytic section $\zeta(s)$ of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$ such that

$$Y(s) = Y + \Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) + \zeta(s)$$

and hence the equation $Y(s) = Y(p)$ is equivalent to

$$(2.6) \quad \Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) + \zeta(s) = 0.$$

Equation (2.6) implies that, near p on the zero locus of ν ,

$$(2.7) \quad \Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$$

on the zero locus of ν . Conversely, whenever equation (2.7) holds, $Y = Y(p)$ is a real grading of W which preserves $F(s)$. Because these two properties specify $Y(s)$ uniquely, it then follows that whenever equation (2.7) holds, $Y(s) = Y(p)$. Thus, on a neighborhood of p , the zero locus of ν is given by equation (2.7).

Applying $e^{-\text{ad } \Gamma(s)}$ to both sides of (2.7), it then follows that the equation for the zero locus near p is

$$(2.8) \quad e^{-\text{ad } \Gamma(s)}\Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) \in \mathfrak{g}_{\mathbb{C}}^F \cap W_{-1}\mathfrak{g}_{\mathbb{C}}.$$

However, q_F is a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$ which is a vector space complement to $\mathfrak{g}_{\mathbb{C}}^F$ in $\mathfrak{g}_{\mathbb{C}}$. Furthermore, q_F is closed under $\text{ad } Y$. Therefore, $\Gamma(s)$, $\Gamma_0(s)$ and $\Gamma_{-1}(s)$ take values in q_F , and $e^{-\text{ad } \Gamma(s)}\Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s)$ takes values in q_F . Consequently, equation (2.8) is equivalent to

$$(2.9) \quad e^{-\text{ad } \Gamma(s)}\Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) = 0.$$

Thus, since $e^{-\text{ad } \Gamma(s)}$ is invertible, equation (2.9) implies the following result.

Theorem 2.10. *Near p , the zero locus of ν is given by the equation $\Gamma_{-1}(s) = 0$.*

Proof. Applying $e^{\text{ad } \Gamma(s)}$ to (2.9) implies that the zero locus is given by the equation

$$(2.11) \quad \Psi(\text{ad } \Gamma_0(s))\Gamma_{-1}(s) = 0$$

By the Campbell–Baker–Hausdorff formula,

$$\Psi(u)v = v + \sum_{j>0} c_j(\text{ad } u)^j v$$

and hence

$$(2.12) \quad \Psi(\text{ad } \Gamma_0)\Gamma_{-1} = \Gamma_{-1} + \sum_{j>0} c_j(\text{ad } \Gamma_0)^j \Gamma_{-1}.$$

Consequently, if

$$\Gamma_0 = \sum_{k>0} \Gamma^{-k,k}, \quad \Gamma_{-1} = \sum_{\ell>0} \Gamma^{-\ell,\ell-1}$$

denote the decomposition of Γ_0 and Γ_{-1} into Hodge components with respect to the bigrading (2.5) then

$$\Psi(\text{ad } \Gamma_0)\Gamma_{-1} = \Gamma^{-1,0} \mod \bigoplus_{r \geq 2} \mathfrak{g}^{-r,r-1}.$$

As such, the equation (2.11) then implies that $\Gamma^{-1,0} = 0$. Proceeding by induction, assume that $\Gamma^{-\ell,1-\ell} = 0$ for $\ell < n$. Then,

$$\Psi(\text{ad } \Gamma_0)\Gamma_{-1} = \Gamma^{-n,n-1} \mod \bigoplus_{r \geq n+1} \mathfrak{g}^{-r,r-1}$$

and hence the equation (2.11), $\Gamma^{-n,n-1} = 0$. Thus, $\Gamma_{-1} = 0$ is the local defining equation for \mathcal{Z} . \square

3. LIMITING GRADING

In this section, we prove that when S is a curve, the grading (2.2) has a well defined limit Y^\dagger as s approaches a puncture $p \in S$. Simple examples show that in higher dimensions, the limiting value of (2.2) depends not only on the point in the boundary divisor but also the direction of approach.

Let $\Delta \subset S$ be a disk containing the puncture p . By passing to a finite cover if necessary, we can assume that the local monodromy of the restriction of \mathcal{V} to the punctured disk $\Delta^* = \Delta - \{p\}$ is unipotent. Let s be a local coordinate on Δ which vanishes at p , let A be an angular sector of Δ^* and s_o be a point in A . Then, we can parallel translate the Hodge filtration of \mathcal{V} back to a single valued filtration $F(s)$ on $V = \mathcal{V}_{s_o}$. Analytic continuation of $F(s)$ to all of Δ^* then gives the multivalued period map

$$\varphi : \Delta^* \rightarrow \Gamma \backslash \mathcal{M}$$

of \mathcal{V} . By local liftability, there exists a holomorphic, horizontal lifting of φ to a map \tilde{F} from the upper half-plane U into \mathcal{M} which makes the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{\tilde{F}} & \mathcal{M} \\ e^{2\pi iz} \downarrow & & \downarrow \\ \Delta^* & \xrightarrow{\varphi} & \Gamma \backslash \mathcal{M} \end{array}$$

Furthermore, upon picking a branch of $\log(s)$ on A and letting

$$z = x + iy = \frac{1}{2\pi i} \log(s)$$

there is unique lifting $\tilde{F}(z)$ such that, for $s \in A$,

$$\tilde{F}(z) = F(s).$$

By unipotent monodromy, we $\tilde{F}(z+1) = e^N \cdot \tilde{F}(z)$ and hence

$$\tilde{\varphi}(z) = e^{-zN} \cdot \tilde{F}(z)$$

drops to a map $\tilde{\varphi}$ from Δ^* into the “compact dual” of \mathcal{M} . The admissibility of \mathcal{V} then asserts that

- (a) $F_\infty = \lim_{s \rightarrow 0} \tilde{\varphi}(s)$ exists;
- (b) The relative weight filtration M of W and N exists.

From these properties, together with Schmid’s nilpotent orbit theorem, Deligne then deduces [9] that the pair (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1, -1)$ -morphism.

The mixed Hodge structure (F_∞, M) induces a mixed Hodge structure on $\mathfrak{g}_\mathbb{C}$ with Deligne bigrading

$$(3.1) \quad \mathfrak{g}_\mathbb{C} = \bigoplus_{r,s} \mathfrak{g}^{r,s}.$$

Note that in equation (3.1), it is possible to have $r + s > 0$ since elements of $\mathfrak{g}_\mathbb{C}$ only preserve W and not M . Nonetheless, the nilpotent subalgebra

$$q_\infty = \bigoplus_{r < 0} \mathfrak{g}^{r,s}$$

is a vector space complement to $\mathfrak{g}_\mathbb{C}^{F_\infty}$. Reasoning as in §2 (cf. [P1]), it then follows that near the puncture $s = 0$ we can write $\tilde{\varphi}(s) = e^{\Gamma(s)}.F_\infty$ relative to a unique holomorphic function $\Gamma(s)$ which takes values in q_∞ and vanishes at $s = 0$. Untwisting the definition of $\tilde{\varphi}$, it then follows that

$$(3.2) \quad F(s) = e^{\frac{1}{2\pi i} \log(s)N} e^{\Gamma(s)}.F_\infty$$

over the angular sector A .

To determine the asymptotic behavior of the grading

$$Y(s) = Y_{(F(s), W)}$$

on A we shall use equation (3.2) together with the SL_2 -orbit theorem of [7] and a result of Deligne which constructs a grading Y of the weight filtration W which is well adapted to both N and the limiting mixed Hodge structure (F_∞, M) .

More precisely, suppose that Y_M is a grading of M which preserves W and satisfies

$$[Y_M, N] = -2N.$$

Then, Deligne (see [4, Appendix, Theorem 1]) shows that there exists a unique, functorial grading

$$(3.3) \quad Y' = Y'(N, Y_M)$$

such that Y' commutes with both N and Y_M ². Furthermore,

- (a) If Y_M is defined over \mathbb{R} then so is Y' ;
- (b) If (F, M) is a mixed Hodge for which N is a $(-1, -1)$ -morphism and induces sub mixed Hodge structures on W then the grading Y' produced from N and the grading of M by the $I^{p,q}$'s of (F, M) preserves F .

²The general statement [6] of Deligne's result for longer weight filtrations involves the interplay between the decomposition of N according to $\text{ad } Y'$ and the graded representations of sl_2 .

To show the existence of the limiting grading

$$Y^\dagger = \lim_{s \rightarrow 0} Y(s)$$

we now invoke the SL_2 -orbit theorem of [2, Proposition 2.20]: Let

$$(\hat{F}, M) = (e^{i\delta}.F_\infty, M)$$

denote Deligne's splitting of the limiting mixed Hodge structure of \mathcal{V} and

$$\Lambda^{-1,-1} = \bigoplus_{r,s < 0} \mathfrak{g}_{(\hat{F}, M)}^{r,s}.$$

Define $G_{\mathbb{R}} = G_{\mathbb{C}} \cap \mathrm{GL}(V_{\mathbb{R}})$ and let $\mathfrak{g}_{\mathbb{R}}$ denote the Lie algebra of $G_{\mathbb{R}}$. Then, there exists a distinguished, real analytic function

$$g : (a, \infty) : (a, \infty) \rightarrow G_{\mathbb{R}}$$

and element

$$\zeta \in \mathfrak{g}_{\mathbb{R}} \cap \ker(N) \cap \Lambda^{-1,-1}$$

such that

- (a) $e^{iyN}.F_\infty = g(y)e^{iyN}.\hat{F}$;
- (b) $g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form

$$\begin{aligned} g(y) &= e^\zeta (1 + g_1 y^{-1} + g_2 y^{-2} + \dots) \\ g^{-1}(y) &= (1 + f_1 y^{-1} + f_2 y^{-2} + \dots) e^{-\zeta} \end{aligned}$$

with $g_k, f_k \in \ker(\mathrm{ad} N)^{k+1}$;

- (c) δ, ζ and the coefficients g_k are related by the formula

$$e^{i\delta} = e^\zeta \left(1 + \sum_{k>0} \frac{1}{k!} (-i)^k (\mathrm{ad} N_0)^k g_k \right).$$

Combining this result with equation (3.2), we obtain the following asymptotic formula for $Y(s)$ over the angular sector A :

$$\begin{aligned} F(s) &= e^{zN} e^{\Gamma(s)}.F_\infty = e^{xN} e^{\Gamma_1(s)} e^{iyN}.F_\infty \\ &= e^{xN} e^{\Gamma_1(s)} g(y) e^{iyN}.\hat{F} = e^{xN} g(y) e^{\Gamma_2(s)} e^{iyN}.\hat{F} \end{aligned}$$

where $\Gamma_1(s) = \mathrm{Ad}(e^{iyN})\Gamma(s)$ and $\Gamma_2(s) = \mathrm{Ad}(g^{-1}(y))\Gamma_1(s)$.

Let \hat{Y}_M denote the grading of M defined by the $I^{p,q}$'s of (\hat{F}, M) and \hat{Y} be the grading of W defined by application of Deligne's construction to the pair (N, \hat{Y}_M) . Then [7],

$$H = \hat{Y}_M - \hat{Y}$$

belongs to $\mathfrak{g}_{\mathbb{R}}$ and satisfies $[H, N] = -2N$. Furthermore, since \hat{Y}_M and \hat{Y} preserve \hat{F} , so does H . Therefore,

$$e^{iyN}.\hat{F} = y^{-\frac{1}{2}H}.F_o$$

where $F_o = e^{iN}.\hat{F}$. By the SL_2 -orbit theorem, F_o belongs to \mathcal{M} . Consequently,

$$F(s) = e^{xN}g(y)e^{\Gamma_2(s)}y^{-\frac{1}{2}H}.F_o = e^{xN}g(y)y^{-\frac{1}{2}H}e^{\Gamma_3(s)}.F_o$$

where

$$\Gamma_3(s) = \text{Ad}(y^{\frac{1}{2}H})\Gamma_2(s) = \text{Ad}(y^{\frac{1}{2}H}g(y)e^{iyN})\Gamma(s).$$

To continue, observe that since, $y = -\frac{1}{2\pi}\log|s|$ and H has only finitely many eigenvalues (all of which are integral), the action of $\text{Ad}(y^{\frac{1}{2}H})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by an integral power of $y^{\frac{1}{2}}$. Similarly, since $g(y)$ is bounded as $s \rightarrow 0$, so is the action of $\text{Ad}(g(y))$. Likewise, since N is nilpotent, the action of $\text{Ad}(e^{iyN})$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by a power of y . Therefore, since $\Gamma(s)$ is a holomorphic function of s which vanishes at $s = 0$, $\Gamma_3(s)$ is a real analytic function on A which satisfies the growth condition

$$\Gamma_3(s) = O((\log|s|)^b s)$$

for some half integral power b . In particular, near $s = 0$,

$$Y_{(e^{\Gamma_3(s)}.F_o, W)} = Y_{(F_o, W)} + \gamma_4(s)$$

for some real analytic function $\gamma_4(s)$ which is again of order $(\log|s|)^b s$. By Deligne [2],

$$Y_{(F_o, W)} = \hat{Y}.$$

Therefore,

$$\begin{aligned} Y(s) &= e^{xN}g(y)y^{-\frac{1}{2}H}.Y_{(e^{\Gamma_3(s)}.F_o, W)} \\ &= e^{xN}g(y)y^{-\frac{1}{2}H}.(Y + \gamma_4(s)) \\ &= e^{xN}g(y).(\hat{Y} + \gamma_5(s)) \end{aligned}$$

where $\gamma_5(s) = \text{Ad}(y^{-\frac{1}{2}H})\gamma_4(s)$ is again of order $\log|s|^{b'}s$ for some half-integral power b' .

Define

$$\begin{aligned} \tilde{g}(s) &= e^{xN}g(y)e^{-xN} \\ &= e^{\zeta} \left(1 + \sum_{k>0} (\text{Ad}(e^{xN}g_k))y^{-k} \right). \end{aligned}$$

Then, since $x = \frac{1}{2\pi}\text{Arg}(s)$ is bounded on the angular sector A ,

$$\lim_{s \rightarrow 0} g(s) = e^{\zeta}.$$

Consequently, because N commutes with \hat{Y} ,

$$(3.4) \quad Y(s) = \tilde{g}(s).(\hat{Y} + \text{Ad}(e^{xN})\gamma_5(s)).$$

Therefore, since $\gamma_5(s)$ is order $(\log(s))^{b'}s$, we can take the limit of equation (3.4) to obtain:

Theorem 3.5.

$$(3.6) \quad Y^\dagger = \lim_{s \rightarrow 0} Y(s) = e^\zeta.\hat{Y}.$$

Remark 3.7. Since the right hand side of (3.6) depends only on the triple (F_∞, W, N) , Y^\dagger is independent of choice of angular sector A . Likewise, a change of local coordinate s changes F_∞ to $e^{\lambda N}.F_\infty$. Therefore, due to the functorial nature of Deligne's construction of the grading Y' and the fact that $[Y', N] = 0$, the right hand side of (3.6) is independent of the choice of coordinate s . Likewise, since the right hand side of (3.6) commutes with N , it is well defined independent of the choice of reference fiber. Consequently, in the geometric setting, Y^\dagger should have a direct geometric meaning.

4. ZERO LOCUS AT INFINITY

To verify the conjecture in the case where S is a curve, we now note that the finiteness condition $(*)$ is preserved under passage to finite covers. Therefore, we may assume as in §3 that the associated variation of mixed Hodge structure \mathcal{V} has unipotent monodromy about each point $p \in D$. The requirement that the zero locus of ν has only finitely many irreducible components on a neighborhood of $p \in D$ is then equivalent to the existence of a disk $\Delta \subset S$ such that $\Delta \cap D = \{p\}$ on which the zero locus of ν is either

- (a) The empty set;
- (b) All of Δ , in which case \mathcal{V} is the trivial extension of $\mathbb{Z}(0)$ by \mathcal{H} .

Applying Deligne's construction (3.3) to the limiting mixed Hodge structure (F_∞, M) , we get a grading Y_∞ of W which preserves F_∞ . Therefore,

$$Y_\infty(s) = e^{\frac{1}{2\pi i} \log(s)N} e^{\Gamma(s)}.Y_\infty$$

is a (complex) grading of W which preserves the Hodge filtration of $F(s)$ near $s = 0$ over the angular sector A . Decomposing $\Gamma(s)$ as

$$\Gamma(s) = \Gamma_0(s) + \Gamma_{-1}(s)$$

according to the eigenvalues of $\text{ad } Y_\infty$, it then follows that

$$\begin{aligned}
 Y_\infty(s) &= e^{\frac{1}{2\pi i} \log(s) N} e^{\Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s)} Y_\infty \\
 &= e^{\frac{1}{2\pi i} \log(s) N} (Y_\infty + \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s)) \\
 (4.1) \quad &= Y_\infty + e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s).
 \end{aligned}$$

As in the derivation of equation (2.11), we then have

$$(4.2) \quad Y(s) = Y_\infty(s) + \zeta(s)$$

for some section $\zeta(s)$ of $W_{-1} \mathfrak{g}_\mathbb{C} \cap \mathfrak{g}_\mathbb{C}^{F(s)}$.

Now, unlike a normal function over Δ^n considered earlier, the function $\zeta(s)$ defined by equation (4.2) may in principle have singularities at $s = 0$. To show that this is not the case, observe that since $\Gamma(s)$ is holomorphic and vanishes at $s = 0$ and N is nilpotent,

$$(4.3) \quad \lim_{s \rightarrow 0} e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) = 0.$$

Therefore, since the limit

$$Y^\dagger = \lim_{s \rightarrow 0} Y(s)$$

exists by Theorem (3.5), equations (4.2) and (4.3) imply that $\zeta(s)$ also has a continuous extension to 0 in the angular sector A .

In particular, if Y^\dagger is not an integral grading of W then there is a neighborhood of zero in angular sector A on which $Y(s)$ is not integral, and hence ν has no zeros on this neighborhood. Thus, it remains to consider the case where Y^\dagger is integral. By [7],

$$\hat{Y} = e^{-i\delta} Y_\infty$$

and hence

$$Y^\dagger = e^\zeta \hat{Y} = e^\zeta e^{-i\delta} Y_\infty.$$

We can write

$$e^\zeta e^{-i\delta} = e^\xi$$

for some (unique)

$$\xi \in \ker(\text{ad } N) \cap \Lambda_{(\hat{F}, M)}^{-1, -1}$$

since both ζ and δ belong to the subalgebra $\ker(N) \cap \Lambda_{(\hat{F}, M)}^{-1, -1}$.

To continue, note that

$$\mathfrak{g}_{(F_\infty, W)}^{r, s} = e^{i \text{ad } \delta} (\mathfrak{g}_{(\hat{F}, M)}^{r, s})$$

and hence

$$\Lambda_{(F_\infty, W)}^{-1, -1} = e^{i \text{ad } \delta} \Lambda_{(\hat{F}, M)}^{-1, -1} = \Lambda_{(\hat{F}, M)}^{-1, -1}$$

since $\Lambda_{(\hat{F}, M)}^{-1, -1}$ is closed under $\text{ad } \delta$. As such

$$\xi \in \ker(\text{ad } N) \Lambda_{(\hat{F}, M)}^{-1, -1} = \ker(\text{ad } N) \cap \Lambda_{(F_\infty, M)}^{-1, -1}.$$

Consequently,

$$Y^\dagger = e^\xi \cdot Y_\infty = Y_\infty + \Psi(\text{ad } \xi_0) \xi_{-1}$$

where ξ_0 and ξ_{-1} both belong to $\ker(\text{ad } N) \cap \Lambda_{(F_\infty, M)}^{-1, -1}$ since Y_∞ commutes with N and preserves each summand $\mathfrak{g}_{(F_\infty, M)}^{r, s}$ of $\Lambda_{(F_\infty, M)}^{-1, -1}$. As such,

$$\Psi(\text{ad } \xi_0) \xi_{-1} \in \ker(\text{ad } N) \cap \Lambda_{(F_\infty, M)}^{-1, -1}.$$

Returning now to equation (4.2), it then follows that

$$Y(s) = Y^\dagger - \Psi(\text{ad } \xi_0) \xi_{-1} + e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) + \zeta(s)$$

where $\zeta(s)$ is a real analytic section of $\mathfrak{g}_{\mathbb{C}}^{F(s)} \cap W_{-1} \mathfrak{g}_{\mathbb{C}}$. In particular, since

$$\lim_{s \rightarrow 0} Y(s) = Y^\dagger$$

is integral, it then follows from the continuity of $Y(s)$ that near $s = 0$ the zeros of ν occur where

$$-\Psi(\text{ad } \xi_0) \xi_{-1} + e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) + \zeta(s) = 0.$$

Equivalently,

$$\begin{aligned} & \text{Ad}(e^{\frac{1}{2\pi i} \log(s) N} e^{\Gamma(s)})^{-1} \left(\Psi(\text{ad } \xi_0) \xi_{-1} - e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) \right) \\ (4.4) \quad & = \text{Ad}(e^{\frac{1}{2\pi i} \log(s) N} e^{\Gamma(s)})^{-1} \zeta(s). \end{aligned}$$

Thus, since the right hand side of equation (4.4) takes values in $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ and the left hand side of (4.4) takes values in q_∞ , it then follows that the zeros of ν occur exactly where

$$e^{\frac{1}{2\pi i} \log(s) \text{ad } N} \Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) = \Psi(\text{ad } \xi_0) \xi_{-1}.$$

Since $\Psi(\text{ad } \xi_0) \xi_{-1} \in \ker(\text{ad } N)$, the equation can be further reduced to just

$$\Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) = -\Psi(\text{ad } \xi_0) \xi_{-1}.$$

Because $\Gamma(s)$ is a holomorphic function which vanishes at zero, the above equation only has solutions near $s = 0$ only if

$$\Psi(\text{ad } \xi_0) \xi_{-1} = 0$$

(i.e. $Y^\dagger = Y_\infty$). In this case, the equation is just

$$\Psi(\text{ad } \Gamma_0(s)) \Gamma_{-1}(s) = 0.$$

Again, because $\Gamma(s)$ is holomorphic at $s = 0$, the solutions to the above equation are either isolated or all of A .

Thus, we have obtained the following.

Theorem 4.5. *Let ν be an admissible normal function on a complex, projective curve S smooth outside of a finite set $D \subset S$. Then the zero locus \mathcal{Z} of ν is an algebraic subset of $S - D$.*

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